Lorentzian worldlines and Schwarzian derivative

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The aim of this note is to relate the classical Schwarzian derivative and the geometry of Lorentz surfaces of constant curvature.

1. The starting point of our investigations lies in the following remark (joint work with L. Guieu). Consider a curve $y = f(x)$ in the Lorentz plane with metric $\mathrm{g} = dxdy$. If $f'(x) > 0$, then its Lorentz curvature can be computed : $\varrho(x) = f''(x)\,(f'(x))^{-3/2}$ and enjoys the quite remarkable property :

$$\sqrt{f'(x)}\,\varrho'(x) = S(f)(x) \tag{1}$$

where

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \tag{2}$$

stands for the Schwarzian derivative of f. (It is well-known that the Schwarzian derivative actually defines a quadratic differential we will write $S(f) = S(f)(x)\,dx^2$.)

2. It is now natural to look for all Lorentz metrics admitting this specific property. We first consider an orientation preserving diffeomorphism $f : \mathbb{R}P^1 \to \mathbb{R}P^1$ whose graph is, therefore, a time-like curve of $\mathbb{R}P^1 \times \mathbb{R}P^1$ endowed with the Lorentz metric $\mathrm{g} = \mathrm{g}(x,y)dxdy$, where $\mathrm{g}(x,y)$ is a positive function. Denoting by t the Lorentz arc-length (also called proper time), we have

Theorem 1 *The necessary and sufficient condition for which the equation*

$$d\varrho\,dt = S(f) \tag{3}$$

holds true for any orientation preserving diffeomorphism f of $\mathbb{R}P^1$ is

$$\mathrm{g} = \frac{dxdy}{(axy + bx + cy + d)^2} \tag{4}$$

where a, b, c, d are arbitrary real constants.

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Note that equation (3) is the intrinsic form of (1). The metric (4) is actually defined on the complement Σ of the graph of the linear-fractional transformation $y = -(bx+d)/(ax+c)$ associated with the singular set of the metric. Clearly, Σ has the topology of a cylinder $\mathbb{R} \times \mathbb{T}$.

3. The scalar curvature of the metric (4) is constant, $R = 8(ad - bc)$. It is well-known [9, 6] that any Lorentz metric of constant curvature can be locally brought into the following forms :

$$\mathrm{g} = dxdy \qquad \text{if } R = 0, \tag{5}$$

$$\mathrm{g} = \frac{8}{R}\frac{dxdy}{(x-y)^2} \quad \text{if } R \neq 0. \tag{6}$$

In our case (4), this equivalence is global and can be obtained by the action of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{R}P^1 \times \mathbb{R}P^1$.

4. *Proof of Theorem 1.* Recall that the curvature of a curve on a (pseudo-)Riemannian surface (Σ, g) is given by $\varrho = \omega(v,a)\,\mathrm{g}(v,v)^{-3/2}$, where v stands for the velocity and $a = \nabla_v v$ for the acceleration vector (ω is the surface 2-form associated with g and ∇ the Levi-Civita connection). For a time-like curve $\tau \mapsto (x(\tau), y(\tau))$ of $\mathbb{R}P^1 \times \mathbb{R}P^1$ one has

$$\varrho = \frac{x'y'' - x''y'}{\mathrm{g}^{1/2}(x'y')^{3/2}} - \frac{x'\partial_x \mathrm{g} - y'\partial_y \mathrm{g}}{\mathrm{g}^{3/2}(x'y')^{1/2}}$$

as a function of the parameter τ. One then easily finds

$$\begin{aligned} \sqrt{\mathrm{g}(v,v)}\,\varrho' &= -\frac{x'''}{x'} + \frac{3}{2}\left(\frac{x''}{x'}\right)^2 + \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 \\ &\quad -x'^2\left[\frac{\partial_x^2 \mathrm{g}}{\mathrm{g}} - \frac{3}{2}\left(\frac{\partial_x \mathrm{g}}{\mathrm{g}}\right)^2\right] + y'^2\left[\frac{\partial_y^2 \mathrm{g}}{\mathrm{g}} - \frac{3}{2}\left(\frac{\partial_y \mathrm{g}}{\mathrm{g}}\right)^2\right]. \end{aligned} \tag{7}$$

In the r.h.s. of equation (7) we recognize the difference $S(y)(\tau) - S(x)(\tau)$ of the Schwarzian derivatives.

Let us show that the extra terms vanish simultaneously iff the metric is given by (4). Indeed, if $\mathrm{g}(x,y) = \partial_x\varphi(x,y) = \partial_y\tilde{\varphi}(x,y)$ with $S(\varphi)(x) = S(\tilde{\varphi})(y) = 0$, then $\varphi(x,y) = (\alpha(y)x + \beta(y))/(\gamma(y)x + \delta(y))$ and $\tilde{\varphi}(x,y) = (\tilde{\alpha}(x)y + \tilde{\beta}(x))/(\tilde{\gamma}(x)y + \tilde{\delta}(x))$ with the unimodularity condition : $\alpha\delta - \beta\gamma = \tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma} = 1$. Since $\partial_x\varphi(x,y) = 1/(\gamma(y)x + \delta(y))^2 = \partial_y\tilde{\varphi}(x,y) = 1/(\tilde{\gamma}(x)y + \tilde{\delta}(x))^2$, the functions $\gamma, \delta, \tilde{\gamma}$ and $\tilde{\delta}$ turn out to be affine, whence (4).

Putting now $y = f(x)$ and $\tau = x$, and using the definition of the arc-length : $\mathrm{g}(v,v) = f'(x)(dx/dt)^2 = 1$, we readily get (3) from (7). ∎

5. Amazingly, our standpoint allows us to recover the definition [4] of the relative Schwarzian derivative of two mappings of $\mathbb{R}P^1$.

Corollary 2 *Given a curve* $\tau \mapsto (x(\tau), y(\tau))$ *of* $\mathbb{R}P^1 \times \mathbb{R}P^1$ *as in Theorem 1, the equation (3) takes the form :*

$$d\varrho\, dt = S(x,y) \tag{8}$$

where $S(x,y)$ *denotes the relative Schwarzian derivative of* x *and* y*.*

6. Recall that in the classical Liouville theory associated with Riemannian surfaces of constant curvature, the Schwarzian derivative enters naturally the transformation law of the Kälher metric under conformal transformations (see, e.g., [1] p. 118).

An analogous phenomenon occurs in the (real) Lorentz framework where the Schwarzian derivative (2) of a conformal diffeomorphism $f \in \mathrm{Conf}(\Sigma, \mathrm{g}) \cong \mathrm{Diff}(\mathbb{R}P^1)$ for the metric (6) is interpreted in [5] as the obstruction for f to be an isometry. The conformal classes of the metrics (5,6) are studied in [2] where they are shown to be symplectomorphic to coadjoint orbits of the Virasoro group. In these papers the Schwarzian derivative appears as the 1-cocycle on the group $\mathrm{Conf}(\Sigma, \mathrm{g})$ and encodes the behavior of the metric near the conformal boundary.

Let us, however, emphasize that our results have no direct relationship with the previous ones.

7. Recently, E. Ghys proved that *given a diffeomorphism* f *of* $\mathbb{R}P^1$*, the Schwarzian derivative* $S(f)$ *has at least four distinct zeroes* [3] (see also [7, 8]). This theorem was discovered as an analogue of the classical four-vertex theorem : *any smooth closed convex plane curve has at least four distinct curvature extrema.* We have thus proved that the Ghys theorem is precisely the four-vertex theorem for time-like closed curves in $\Sigma \subset \mathbb{R}P^1 \times \mathbb{R}P^1$ endowed with the constant curvature metric (4).

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